DETERMINATION OF THE THERMOPHYSICAL CHARACTERISTICS OF ALTERNATIVELY FREEZING AND THAWING DISPERSED MEDIA BY THE METHOD OF SOLVING INVERSE PROBLEMS OF HEAT CONDUCTION

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The article explains an algorithm for determining the thermophysical characteristics of dispersed media with phase transitions based on the method of solving inverse problems of heat conduction.

The problems of constructing and operating various civil-engineering constructions in permafrost regions require investigations for determining the thermophysical characteristics of alternately freezing and thawing soils.

The finding of these values can be formulated in the form of the inverse problem of heat conduction, which has aroused great interest in recent years (see, e.g., [1-3]). The authors of [1, 2] and of the works quoted by them examine problems of reconstructing unknown boundary conditions; Cannon and Duchateau [3] examine the problem of determining the thermophysical characteristics c, λ when they are correlated by $\lambda(T) = kc(T)$, where $k = \text{const} > 0$.

We examined the numerical determination of c, λ for problems with phase transitions and without them. For the sake of simplicity, we explain the problem of freezing in the absence of free moisture, although the method is also being extended to more general problems of freezing in the temperature spectrum.

The heat flux $q(\tau)$ is supplied to the cylindrical surface of a specimen with the initial temperature T_0 and moisture ω_0. The change in the temperature field is described by the equation

$$c(T)\rho\frac{\partial T}{\partial \tau}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\lambda(T)\frac{\partial T}{\partial r}\right), \tag{1}$$

where

$$c(T)=c_{sk}+c_i\omega_0+(c_w-c_i)\,\omega(T)+L\frac{\partial\omega(T)}{\partial T}.$$

The function $\omega(T)$ at the point of start of freezing is equal to the bound moisture ω_0. With the lowering of the temperature, $\omega(T)$ decreases continuously, and beginning at some value $T = T_1$, it remains constant if the specimen contains strongly bound moisture $\omega(T) = \omega_{sb}$. For most of the typical dispersed soils [4-6] (clay, loam, sand) the dependence of the amount of nonfrozen water on the temperature in the range $T_1 \leqslant T \leqslant 0$ can be approximated by a hyperbolic arc. Thus, the function $\omega(T)$ can be represented in the form

$$\omega(T)=\begin{cases}\omega_{sb}, & T\leqslant T_1,\\ \dfrac{a}{|T+T_2|^b} & T_1\leqslant T\leqslant 0,\\ \omega_0, & T\geqslant 0.\end{cases} \tag{2}$$

Corresponding to this, the volume heat capacity for the three mentioned zones has the form

$$c(T)=\begin{cases}c_{sk}+c_i\omega_0+(c_w-c_i)\,\omega_{sb}=c_f, & T\leqslant T_1,\\ c_{sk}+c_i\omega_0+(c_w-c_i)\,\omega(T)+L\dfrac{\partial\omega(T)}{\partial T}, & T_1\leqslant T\leqslant 0,\\ c_{sk}+c_w\omega_0=c_m, & T\geqslant 0,\end{cases} \tag{3}$$

$\lambda(T)$ is the thermal conductivity whose temperature dependence can be expressed in the form [6]

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$$\lambda(T)=\begin{cases}\lambda_f, & T\leqslant T_1,\\ \lambda_f-(\lambda_f-\lambda_m)\dfrac{\omega(T)-\omega_{sb}}{\omega_0-\omega_{sb}}, & T_1\leqslant T\leqslant 0,\\ \lambda_m, & T\geqslant 0.\end{cases} \tag{4}$$

Joined to Eq. (l) are the initial and boundary conditions:

$$T(r,\ 0)=T_0,\quad 0\leqslant r\leqslant R, \tag{5}$$

$$\lambda\frac{\partial T}{\partial r}=0,\quad r=0,\quad 0\leqslant\tau\leqslant\Theta, \tag{6}$$

$$\lambda\frac{\partial T}{\partial r}=q(\tau),\quad r=R,\quad 0\leqslant\tau\leqslant\Theta. \tag{7}$$

In addition to conditions (5)-(7), the temperatures at the center and on the surface, $U_1(\tau)$, $U_2(\tau)$, respectively, are measured. One of the possible methods of solving such problems is treating them in the form of problems of optimum control, i.e., select the control z = (c, λ) from the condition of certain agreement of the measured temperatures $U_1(\tau)$, $U_2(\tau)$ with the calculated temperatures corresponding to the given controls. These conditions are usually formulated in the form of a problem of minimizing some functional

$$I(z)=\int_0^\Theta\{\beta_1[T(0,\ \tau,\ z)-U_1(\tau)]^2+\beta_2[T(R,\ \tau,\ z)-U_2(\tau)]^2\}\,d\tau, \tag{8}$$

where $\beta_1(\tau)$, $\beta_2(\tau)$ are specified positive functions characterizing the extent of confidence of the supplementary information.

Let us examine the difference analog of the inverse problem. On the sections $0\leqslant r\leqslant R$, $0\leqslant\tau\leqslant\Theta$ we introduce grids (for the sake of simplicity of the presentation they are uniform) with the nodes $r_i = ih$, $i = 0, 1, \ldots, N$, $\tau_j=j\Delta\tau$, $j=0,\ 1,\ \ldots,\ n$, $h=R/N$, $\Delta\tau=\Theta/n$.

The difference analogs of Eqs. (1), (5)-(7), constructed by the balance method, are the equations

$$c_{ij}\rho\frac{T_{ij}-T_{ij-1}}{\Delta\tau}=\frac{1}{r_ih^2}\,[\lambda_{i+0.5j}r_{i+0.5}(T_{i+1j}-T_{ij})\ -\lambda_{i-0.5j}r_{i-0.5}(T_{ij}-T_{i-1j})], \tag{9}$$

$$i=1,\ 2,\ \ldots,\ N-1,\quad j=1,\ 2,\ \ldots,\ n,$$

$$c_{0j}\rho\frac{T_{0j}-T_{0j-1}}{\Delta\tau}=\frac{4\lambda_{0.5j}}{h^2}(T_{1j}-T_{0j}),\ j=1,2,\ \ldots,\ n, \tag{10}$$

$$c_{Nj}\rho\frac{T_{Nj}-T_{Nj-1}}{\Delta\tau}=-\frac{4(2R-h)}{(4R-h)h^2}\lambda_{N-0.5j}(T_{Nj}-T_{N-1j})+\frac{8R}{(4R-h)h}q_j,\ j=1,\ 2,\ \ldots,\ n, \tag{11}$$

$$T_{i0}=T_0,\quad i=0,\ 1,\ \ldots,\ N. \tag{12}$$

We replace the difference problem (9)-(12) for finding the grid functions c_{ij}, $\lambda_{i+0.5j}$ by the external problem of the minimum of the functional

$$I=\sum_{j=1}^{n}[\beta_{1j}(T_{0j}-U_{1j})^2+\beta_{2j}(T_{Nj}-U_{2j})^2]\,\Delta\tau. \tag{13}$$

From the expression of the coefficients of (2), (3), (4) follows that for finding them it is necessary to find c_{sk}, λ_f, λ_m (the parameter b is assumed to be unknown), and also ω_{sb} and the points T_1, T_2. It is difficult to determine all these magnitudes simultaneously. We examined the following sequence of finding the unknowns: from the experimental thermogram the regions of quasi-steady-state regime in frozen and thawed zones are selected, and in each region the unknowns c_f, λ_f, c_m, λ_m are determined. Then, with the aid of (3), c_{sk} and ω_{sb} are calculated, and at the last stage the parameter a and the values of T_1, T_2.

The values c_f, λ_f, c_m, λ_m, a are determined by minimizing the functional (13); the components of its gradient, calculated by the method of [7], have the form

$$\frac{\partial I}{\partial c}=\sum_{j=1}^{n}\sum_{i=0}^{N}\rho\psi_{ij}\frac{T_{ij}-T_{ij-1}}{\Delta\tau}, \tag{14}$$

$$c = c_f \text{ for } T_{ij} \leqslant T_1', \quad c = c_m \text{ for } T_{ij} \geqslant T_2',$$

$$\frac{\partial I}{\partial \lambda} = \sum_{j=1}^{n} \Big\{ \psi_{Nj}(T_{Nj} - T_{N-1j}) \frac{2r_{N-0.5}}{(r_N - 0.25h)h^2} - \psi_{0j}(T_{1j} - T_{0j}) \frac{4}{h^2} - \sum_{i=1}^{N-1} \frac{\psi_{ij}}{r_i h^2} [r_{i+0.5j}(T_{i+1j} - T_{ij}) - r_{i-0.5}(T_{ij} - T_{i-1j})] \Big\}, \tag{15}$$

$$\lambda = \lambda_f \text{ for } T_{ij} \leqslant T_1', \quad \lambda = \lambda_m \text{ for } T_{ij} \geqslant T_2',$$

$$\frac{\partial I}{\partial a} = \begin{cases} 0, & T_{ij} < T_1, \ T_{ij} > 0, \\ \sum_{j=1}^{n} \Big\{ \psi_{Nj}(T_{Nj} - T_{N-1j}) \frac{2a_{N-0.5j}\, r_{N-0.5}}{(r_N - 0.25h)h^2} - \psi_{0j} a_{0.5j}(T_{1j} - T_{0j}) \times \\ \times \frac{4}{h^2} - \sum_{i=1}^{N-1} \frac{\psi_{ij}}{r_i h^2} [r_{i+0.5}\, a_{i+0.5j}(T_{i+1j} - T_{ij}) - \\ - r_{i-0.5}\, a_{i-0.5j}(T_{ij} - T_{i-1j})] + \sum_{i=0}^{N} \rho \psi_{ij} \frac{T_{ij} - T_{ij-1}}{\Delta\tau} \times \\ \times \Big(\frac{c_w - c_i}{|T_{ij} + T_2|^b} + \frac{Lb}{|T_{ij} + T_2|^{b+1}} \Big) \Big\}, & T_1 \leqslant T_{ij} \leqslant 0, \end{cases} \tag{16}$$

where $a_{ij} = \dfrac{\lambda_f - \lambda_m}{\omega_0 - \omega_{sb}} \dfrac{1}{|T_{ij} + T_2|^b}$; $T_1' \leqslant T_1$, $T_2' > 0$ are selected from the experimental thermogram; ψ_{ij} is the solution of the conjugated system.

After having determined the gradients of the functional (13), we can use various gradient-type iteration methods for its minimization. For finding the unknown z, an iteration procedure constructed according to the method of projecting gradients was used:

$$z^{s+1} = P_M(z^s - \alpha_s I'(z^s)). \tag{17}$$

The iteration process of finding the parameters a, T_1, T_2 is organized in the following way. We know a^s, $s \geqslant 0$. Then T_1^s, T_2^s are determined from the following equalities ensuing from (2):

$$T_2^s = -\left(\frac{a^s}{\omega^0}\right)^{1/b}, \quad T_1 = -T_2^s - \left(\frac{a^s}{\omega_{sb}}\right)^{1/b}. \tag{18}$$

Then with the aid of the solutions of the initial and conjugate boundary problems by formula (16), taking the found values of T_1^s, T_2^s into account, we calculate the gradient $\partial I/\partial a$, and from (17) we find the constant a^{s+1}. If we replace s in (18) by s + 1, we obtain T_1^{s+1}, T_2^{s+1}. Then we turn to finding s + 2 of the approximation etc.

To check the effectiveness of the algorithm, calculations were carried out to solve a model problem with the following initial data:

$$T(r, \tau) = a_0(\tau) + a_1(\tau) r^2 + a_2 r^4, \tag{19}$$

$$q(\tau) = q_0 + q_1\tau, \tag{20}$$

where $a_0(\tau) = 16\lambda^2(\tau^2 + 3\tau + 1)/c^2\rho^2$; $a_1(\tau) = 16\lambda(2\tau + 3)/c\rho$; $a_2 = 2$, $q_0 = 8\lambda + 96\lambda^2/c\rho$; $q_1 = 64\lambda^2/c\rho$, $\rho = 15$; $z = (c, \lambda)$; $c = 0.5$; $\lambda = 1$. The problem was solved with accurate values of the supplementary information

$$U_1(\tau_j) = T(0, \tau_j), \quad U_2(\tau_j) = T(R, \tau_j) \tag{21}$$

TABLE 1. Reconstruction of the Thermophysical Characteristics in the Model Problem

s	Without perturbations			With perturbations		
	c^s	λ^s	I^s	c^s	λ^s	I^s
0	0,3	1,6		0,3	1,6	
8	0,4963	1,5204	16,56	0,5165	1,5	19,23
24	0,505	1,2657	5,68	0,5088	1,2118	6,88
56	0,5074	1,1608	2,39	0,5132	1,0778	2,9
96	0,5088	1,1170	1,38	0,5177	0,997	1,85

TABLE 2. Reconstruction of the Thermophysical Characteristics and of the Amount of Nonfrozen Water by Formulas (I), (II), (III) and by the Traditional Method (IV)

Formula	c_{sk}, kJ / kg · deg	λ_f, W /kg · deg	λ_m, W/m · deg	a	I
(I)	1,18	1,78	1,36	2,66	0,69
(II)	1,18	1,78	1,36	3,00	0,45
(III)	1,18	1,78	1,36	4,01	0,625
(IV)	1,015	1,80	1,34	4,40	0,

and with some perturbations specified by the formulas

$$U_\alpha(e,\ \tau_j) = T(e,\ \tau_j) + 0.01 b_1 \sin(b_2 \tau_j)\, T(e,\ \tau_j), \tag{22}$$

where e = 0, R; $b_1 = 7$; $b_2 = 2$; $\alpha = 1, 2$.

The results of the numerical solution of the problem in question, presented in Table 1, show that the suggested algorithm is satisfactorily effective.

This method was used in the investigation of the thermophysical characteristics and amount of nonfrozen water in alternately freezing and thawing dispersed materials.

We examined the following relations concerning the amount of nonfrozen water:

$$\omega(T) = \begin{cases} \dfrac{a}{|T|^b}, & T \leqslant T_2, \\ \omega_0, & T \geqslant T_2, \end{cases} \tag{I}$$

$$\omega(T) = \begin{cases} \dfrac{a}{|T + T_2|^b}, & T \leqslant 0, \\ \omega_0, & T \geqslant 0, \end{cases} \tag{II}$$

$$\omega(T) = \begin{cases} \omega_{sb}, & T \leqslant T_1, \\ \dfrac{a}{|T + T_2|^b}, & T_1 \leqslant T \leqslant 0, \\ \omega_0, & T \geqslant 0. \end{cases} \tag{III}$$

The coefficients obtained as a result of the numerical solution were compared with the coefficients obtained by the traditional method (Table 2).

NOTATION

r, space coordinate; τ, time; T, temperature of the specimen; T_0, initial temperature; c_i, c_w, c_{sk}, specific heat of ice, water, and of the organic-mineral skeleton, respectively; c_f, c_m, λ_f, λ_m, specific heat and thermal conductivity in the frozen and melted zones, respectively; c, effective heat capacity; λ, thermal conductivity; ρ, density; ω_0, ω_{sb}, bound and strongly bound moisture, respectively; $\omega(T)$, amount of nonfrozen water; R, radius of the cylinder; $q(\tau)$, heat flux; I, functional; $U_1(\tau)$, $U_2(\tau)$, measured temperatures of the specimen at the points r = 0 and r = R, respectively, at the instant τ; β_1, β_2, degree of confidence of the supplementary information; Θ, final instant of time; a, b, k, α_s, positive constants; L, specific heat of melting; N, number of grid nodes over space; n, number of grid nodes over time; h, grid step over space; $\Delta\tau$, grid step over time; ψ, solution of the conjugate system; s, number of iteration.

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INERTIA OF MEASUREMENTS WITH "AUXILIARY-WALL" TYPE HEAT METERS

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The article examines the problem of thermal inertia on the basis of an "auxiliary-wall" type heat meter. It demonstrates the boundaries of applicability of the approximate relationship for calculating non-steady-state heat fluxes.

Heat meters of the "auxiliary wall" type are widely used for measuring heat fluxes, and schematically they are often represented in the form of a plate attached to a semibounded body. It was shown in [1] that for measuring non-steady-state heat fluxes with such heat meters, it is necessary to know the temperature gradient $\Delta t(\tau)$ on the sensor with known thickness δ, and also the criterion $\varkappa = \frac{\lambda_2}{\lambda_1}\sqrt{\frac{a_1}{a_2}}$, characterizing the thermophysical properties of the heat meter and the half-space. The same article also presented the theoretical relationships for determining the flux $q(\tau)$ in some special cases ($\varkappa = 0, 1, \infty$). For determining a variable flux, it is necessary in the general case (with arbitrary values of $\varkappa$) to use the relationship (I) whose derivation is presented in Appendix 1:

$$q(\tau) = q'(\tau) + q''(\tau); \tag{1}$$

$$q'(\tau) = \frac{\lambda_1}{\sqrt{\pi a_1 \tau}}\left\{1 + \sum_{n=1}^{\infty}\left[1 + \left(\frac{1-\varkappa}{\varkappa+1}\right)^n\right]\exp\left(-\frac{n^2A^2}{4\tau}\right)\right\}\Delta t(\tau) = K_0(\tau)\,\Delta t(\tau); \tag{2}$$

$$q''(\tau) = \frac{\lambda_1}{\sqrt{2\pi a_1}}\int_0^{\tau}\frac{\Delta t(\tau)-\Delta t(\xi)}{\sqrt{(\tau-\xi)^3}}\left\{1 - \sum_{n=1}^{\infty}\left[1 + \left(\frac{1-\varkappa}{\varkappa+1}\right)^n\right]\frac{n^2A^2 - 2(\tau-\xi)}{2(\tau-\xi)}\exp\left[-\frac{n^2A^2}{4(\tau-\xi)}\right]\right\}d\xi; \tag{3}$$

$$\varkappa = \frac{\lambda_2}{\lambda_1}\sqrt{\frac{a_1}{a_2}}; \quad A = \frac{\delta}{\sqrt{a_1}}.$$

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